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An Itô–Stratonovich formula for Gaussian processes: A Riemann sums approach

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Abstract

The aim of this paper is to establish a change of variable formula for general Gaussian processes whose covariance function satisfies some technical conditions. The stochastic integral is defined in the Stratonovich sense using an approximation by middle point Riemann sums. The change of variable formula is proved by means of a Taylor expansion up to the sixth order, and applying the techniques of Malliavin calculus to show the convergence to zero of the residual terms. The conditions on the covariance function are weak enough to include processes with infinite quadratic variation, and we show that they are satisfied by the bifractional Brownian motion with parameters (H, K) such that $1/6 < HK < 1$, and, in particular, by the fractional Brownian motion with Hurst parameter $H \in (1/6, 1)$.

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1. Introduction

In the past few years, an important effort has been made to develop a stochastic calculus with respect to Gaussian processes beyond the Brownian motion, with a strong emphasis on the fractional Brownian motion (fBm, for short). Since, in general, a Gaussian process is

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not a semimartingale, the classical theory of stochastic integration does not apply. The main approaches recently developed on stochastic Gaussian integration are the following ones.

- *Pathwise integrals:* The trajectories of the fBm with Hurst parameter $H \in (0, 1)$ have finite q-variation for any $q > \frac{1}{H}$. Therefore, by the work of Young [18], the pathwise Riemann–Stieltjes integral $\int_0^T u_s \mathrm{d}B_s^H$ exists if the integrator has trajectories of finite p-variation and $\frac{1}{p} + H > 1$. The recent theory of rough path analysis developed from the pioneering work of Lyons [11] has pushed forward the pathwise approach, with applications to the fBm with parameter $H > \frac{1}{4}$ (see Coutin and Qian [4]). On the other hand, following the work by Zähle [19], one can get explicit formulas for the Riemann–Stieltjes integrals using the classical fractional calculus. This approach has been used by Nualart and Rascanu in [13] to solve stochastic differential equations with respect to the fBm with Hurst parameter $H > \frac{1}{2}$.
- *Stochastic calculus of variations*: In the case of the Brownian motion, the classical Itô integral coincides with the divergence operator in the Wiener space. Then, a natural idea is to use the divergence operator to define a stochastic integral with respect to an arbitrary Gaussian process. This approach was introduced by Decreusefond and Üstünel in [5,6] for the fBm. In these papers an Itô formula is given for the divergence integral and its Stratonovich version. The case of more general Gaussian processes has been studied by, among others, Carmona and Coutin in [2], and Alòs, Mazet and Nualart in [1].
- *The regularization approach:* In [16] and [17] Russo and Vallois introduced a new type of stochastic integrals through integrator smoothing. Although the scope of this theory is more general than Gaussian integration, it has been intensively used to develop the stochastic integration with respect to the fBm by Gradinaru, Nourdin, Russo and Vallois in a series of papers; see [7,9,8]. In [9], an Itô formula for the Stratonovich-symmetric integral with respect to the fBm for $H > 1/6$ is established. Moreover, they introduce a generalized type of symmetric integral that allows them to prove an Itô formula for arbitrary $H \in (0, 1)$. Using the extended divergence operator, Cheridito and Nualart (see [3]) have also found the barrier $1/6$ for the existence of the Stratonovich-symmetric integral with respect to the fBm.

This paper is devoted to proving an Itô formula for the Stratonovich integral with respect to Gaussian processes, focusing on the classical Riemann sums approach. In order to establish this formula we use the techniques of Malliavin calculus, and more precisely, the integration by parts formula. This technique has already been used in [14] to prove a Wick–Itô formula for Gaussian processes. We impose some technical conditions on the covariance function, which are satisfied, for instance, for the fBm with Hurst parameter $H > 1/6$.

The paper is organized as follows. In Section 2 we give some preliminaries on stochastic calculus of variations with respect to Gaussian processes and provide the definitions needed in order to state the main result. Section 3 contains the statement and proof of the main result. In Section 4 we state and prove some technical lemmas using extensively the integration by parts formula for the derivative operator. These lemmas will be useful in computing some expectations related to the powers of the increments of the process in the Taylor expansion. Section 5 contains the convergence results used in the proof of the main theorem. We end the paper by giving an example of a Gaussian process which fulfills the conditions for the Itô formula to be valid. The Gaussian process chosen is the bifractional Brownian motion, a generalization of the fractional Brownian motion. In particular, we prove that the condition on the parameters of this process is the same as the one given in [15], where the so called regularization approach is used. In the case of the fractional Brownian motion the condition on the parameter coincides with the sharper one given in [9], where also the regularization approach is used.

Throughout the paper C will denote a constant that may be different from one equation to another one.

2. Preliminaries and notation

Let $X = \{X_t, t \in [0, T]\}$ be a one-dimensional centered Gaussian process with continuous covariance function $R(s,t) = \mathbb{E}[X_s X_t]$ and variance function $V_t = \mathbb{E}\left[X_t^2\right]$. We will make use of Malliavin calculus with respect to the Gaussian process X. Let us introduce the main ingredients of this technique. We refer the reader to [12] for a more detailed account of this material.

Denote by $\mathcal{E}$ the set of step functions in $[0, T]$. Let $\mathcal{H}$ be the completion of $\mathcal{E}$ with respect to the scalar product $\left\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \right\rangle_{\mathcal{H}} \triangleq R(s,t), 0 \le s, t \le T$. The space $\mathcal{H}$ is isometric to the Gaussian Hilbert space generated by the process X and we let $I_1 : h \mapsto X(h), h \in \mathcal{H}$, denote this isometry. In particular, we have $X_t = X(\mathbf{1}_{[0,t]})$.

Let $\mathcal{H}^{\otimes m}$ denote the mth tensor power of $\mathcal{H}$. We also consider the mth symmetric tensor power of $\mathcal{H}$, which we will denote by $\mathcal{H}^{\odot m}$. If $h_i \in \mathcal{H}$, $1 \le i \le n$, we denote their symmetric tensor product by $h_1 \odot \cdots \odot h_n = \frac{1}{n!} \sum_\sigma h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}$, where σ runs over all permutations of $\{1, \ldots, n\}$.

We denote by $\mathcal{S}$ the set of smooth and cylindrical random variables F which have the form

$$F = f(X(h_1), \ldots, X(h_n)), \tag{2.1}$$

with $h_1, \ldots, h_n \in \mathcal{H}$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (the space of bounded functions which have bounded derivatives of all orders) and $n \in \mathbb{N}$.

If $F \in \mathcal{S}$ is a random variable of the form (2.1) one can define the derivative operator D as follows: $DF = \sum_{i=1}^n \partial_i f(X(h_1), \ldots, X(h_n)) h_i$, which is an element of $L^2(\Omega; \mathcal{H})$. By iteration one can define $D^m F$, which is an element of $L^2\left(\Omega; \mathcal{H}^{\odot m}\right)$.

For $m \ge 1$ and $p \ge 1$ the space $\mathbb{D}^{m,p}$ is the completion of $\mathcal{S}$ with respect to the norm $\|F\|_{m,p}$, where $\|F\|_{m,p}^p = \mathbb{E}[|F|^p] + \sum_{i=1}^m \mathbb{E}[\left\| D^i F \right\|_{\mathcal{H}^{\otimes i}}^p]$.

The following definition will provide the class of functions for which the Itô formula will be valid.

Definition 1. We will say that a function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}_{l\text{-}\exp}^m(\mathbb{R}), m, l \in \mathbb{N}$, if $f \in \mathcal{C}^m(\mathbb{R})$ and f and its derivatives until the order m satisfy the following exponential growth condition:

$$\left| f^{(k)}(x) \right| \le C_T e^{c_T x^2}, \quad k = 0, \ldots, m, \ x \in \mathbb{R}, \tag{2.2}$$

where $C_T > 0$,

$$0 < c_T < \frac{1}{2l} \left(\sup_{t \in [0,T]} V_t \right)^{-1}, \tag{2.3}$$

and V_t is the variance function of the Gaussian process X.

Obviously, if $f \in \mathcal{C}_{l\text{-}\exp}^m(\mathbb{R})$ then $f \in \mathcal{C}_{k\text{-}\exp}^m(\mathbb{R})$, with $k < l$.

We fix a class $\mathcal{D}$ of finite partitions $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of the interval $[0, T]$ such that

$$\sup_{\pi \in \mathcal{D}} \frac{|\pi|}{|\pi|_{\inf}} < \infty, \tag{2.4}$$

and $\inf_{\pi \in \mathcal{D}} |\pi| = 0$, where $|\pi| \triangleq \max_{0 \le i \le n-1} |t_{i+1} - t_i|$ is the partition size and $|\pi|_{\inf} \triangleq \min_{0 \le i \le n-1} |t_{i+1} - t_i|$. Notice that the classes of uniform and dyadic partitions satisfy condition (2.4).

Given a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0, T]$ and a stochastic process $X = \{X_t, t \in [0, T]\}$ we can define the following quantities:

$$\alpha_\pi \triangleq \sup_{i \ge 0} \mathbb{E}[(\Delta_i X)^2], \qquad \beta_\pi \triangleq \sup_{0 \le t \le T} \left(\sum_{i=0}^{n-1} |\mathbb{E}[X_t \Delta_i X]| \right),$$

$$\gamma_\pi \triangleq \sum_{i,j=0}^{n-1} |\mathbb{E}[\Delta_i X \Delta_j X]|,$$

$$\delta_\pi \triangleq \sup_{\substack{0 \le t \le T \\ i \ge 0}} |\mathbb{E}[X_t \Delta_i X]|, \qquad \varepsilon_\pi \triangleq \sum_{i=0}^{n-1} \sup_{0 \le t \le T} |\mathbb{E}[X_t \Delta_i X]|,$$

where $\Delta_i X = X_{t_{i+1}} - X_{t_i}$.

Definition 2. We denote by $\mathcal{C}$ the class of centered Gaussian stochastic processes X satisfying the following conditions:

$$\Lambda_\pi \triangleq \alpha_\pi \left(\alpha_\pi + \delta_\pi^2 \right) (\gamma_\pi + \beta_\pi \varepsilon_\pi) \to 0 \quad \text{as } |\pi| \to 0, \tag{2.5}$$

$$\Psi_\pi \triangleq \delta_\pi^2 \varepsilon_\pi \to 0 \quad \text{as} \quad |\pi| \to 0, \tag{2.6}$$

where π runs over all the partitions in the class $\mathcal{D}$.

Finally, we introduce the Stratonovich integral.

Definition 3. Let $\{u_t : t \in [0, T]\}$ be a stochastic process. The Stratonovich integral $\int_0^T u_t \circ \mathrm{d}X_t$ is defined as the limit in $L^2(\Omega)$, if it exists, of the symmetric Riemann sums $\sum_{i=0}^{n-1} \frac{u_{t_{i+1}} + u_{t_i}}{2} \left(X_{t_{i+1}} - X_{t_i} \right)$ as $|\pi| \to 0$, where $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ runs over all the partitions in $\mathcal{D}$.

3. Itô's formula for the Stratonovich integral

Here is the main result of this paper.

Theorem 4. *Suppose that the Gaussian process X belongs to the class $\mathcal{C}$ and the function f belongs to $\mathcal{C}^{11}_{4-\exp}(\mathbb{R})$. Then the Stratonovich integral $\int_0^T f'(X_s) \circ \mathrm{d}X_s$ exists and the following Itô formula holds: $f(X_T) = f(X_0) + \int_0^T f'(X_s) \circ \mathrm{d}X_s$.*

Proof. Given $a < b \in \mathbb{R}$ and a function f of class $\mathcal{C}^6(\mathbb{R})$, set $I(a, b) = f(b) - f(a) - (b - a) \frac{f'(b) + f'(a)}{2}$. We are going to compute the Taylor expansion of $I(a, b)$ with respect to the point $S \triangleq \frac{a+b}{2}$ up to the sixth order. We have

$$f(b) = f(S) + \sum_{k=1}^{5} \frac{1}{k! 2^k} f^{(k)}(S) (b - a)^k + \frac{1}{6! 2^6} f^{(6)}(\widehat{S}^+) (b - a)^6,$$

$$f(a) = f(S) + \sum_{k=1}^{5} \frac{(-1)^k}{k!2^k} f^{(k)}(S)(b-a)^k + \frac{1}{6!2^6} f^{(6)}(\widehat{S}^-)(b-a)^6,$$

$$f'(b) = \sum_{k=1}^{5} \frac{1}{(k-1)!2^{k-1}} f^{(k)}(S)(b-a)^{k-1} + \frac{1}{5!2^5} f^{(6)}(\widetilde{S}^+)(b-a)^5,$$

$$f'(a) = \sum_{k=1}^{5} \frac{(-1)^{k-1}}{(k-1)!2^{k-1}} f^{(k)}(S)(b-a)^{k-1} + \frac{1}{5!2^5} f^{(6)}(\widetilde{S}^-)(b-a)^5,$$

where $\widehat{S}^+$ and $\widetilde{S}^+$ are two different points belonging to $[S, b]$, and $\widehat{S}^-$ and $\widetilde{S}^-$ are two different points belonging to $[a, S]$. Therefore,

$$\begin{aligned} I(a,b) &= \sum_{k=1}^{5} \left(\frac{1+(-1)^{k+1}}{k!2^k} - \frac{1+(-1)^{k-1}}{(k-1)!2^k} \right) f^{(k)}(S)(b-a)^k \\ &\quad + \left\{ \frac{1}{6!2^6} \left(f^{(6)}(\widehat{S}^+) - f^{(6)}(\widehat{S}^-) \right) - \frac{1}{5!2^6} \left(f^{(6)}(\widetilde{S}^+) + f^{(6)}(\widetilde{S}^-) \right) \right\} (b-a)^6 \\ &= -\frac{1}{12} f^{(3)}(S)(b-a)^3 - \frac{1}{480} f^{(5)}(S)(b-a)^5 + R_6, \end{aligned}$$

where

$$R_6 = \left\{ \frac{1}{6!2^6} \left(f^{(6)}(\widehat{S}^+) - f^{(6)}(\widehat{S}^-) \right) - \frac{1}{5!2^6} \left(f^{(6)}(\widetilde{S}^+) + f^{(6)}(\widetilde{S}^-) \right) \right\} (b-a)^6.$$

Now consider a partition $\pi = \{0 = t_0 < \cdots < t_n = T\}$. One has

$$\begin{aligned} \sum_{i=0}^{n-1} I\left(X_{t_i}, X_{t_{i+1}}\right) &= f(X_T) - f(X_0) - \sum_{i=0}^{n-1} \frac{f'\left(X_{t_{i+1}}\right) + f'\left(X_{t_i}\right)}{2} \Delta_i X \\ &= -\frac{1}{12} \sum_{i=0}^{n-1} f^{(3)}(S_i X)(\Delta_i X)^3 - \frac{1}{480} \sum_{i=0}^{n-1} f^{(5)}(S_i X)(\Delta_i X)^5 + R_6^\pi, \end{aligned}$$

where $\Delta_i X \triangleq X_{t_{i+1}} - X_{t_i}$, $S_i X = (X_{t_{i+1}} + X_{t_i})/2$,

$$R_6^\pi = \sum_{i=0}^{n-1} \left\{ \frac{1}{6!2^6} \left(f^{(6)}(\widehat{S}_i^+) - f^{(6)}(\widehat{S}_i^-) \right) - \frac{1}{5!2^6} \left(f^{(6)}(\widetilde{S}_i^+) + f^{(6)}(\widetilde{S}_i^-) \right) \right\} (\Delta_i X)^6,$$

and $\widehat{S}_i^+$, $\widehat{S}_i^-$, $\widetilde{S}_i^+$ and $\widetilde{S}_i^-$ are defined as before but now using the intervals $[X_{t_i}, S_i X]$ and $[S_i X, X_{t_{i+1}}]$. Define

$$R_3^\pi \triangleq \sum_{i=0}^{n-1} f^{(3)}(S_i X)(\Delta_i X)^3, \qquad R_5^\pi \triangleq \sum_{i=0}^{n-1} f^{(5)}(S_i X)(\Delta_i X)^5.$$

Under the assumptions of the theorem, the terms R_3^π, R_5^π and R_6^π converge in $L^2(\Omega)$ to zero as $|\pi| \to 0, \pi \in \mathcal{D}$, by Propositions 10–12. This implies

$$\lim_{|\pi| \to 0, \pi \in \mathcal{D}} \sum_{i=0}^{n-1} \frac{f'\left(X_{t_{i+1}}\right) + f'\left(X_{t_i}\right)}{2} \Delta_i X = \int_0^T f'(X_s) \circ \mathrm{d}X_s$$

in $L^2(\Omega)$ and hence the theorem is proved. ■

Remark 5. The regularity assumption on f (eleven times continuously differentiable) is required by the method of proof based on the integration by parts formula to control the residual terms, and it cannot be weakened. Also the growth condition (2.2) guarantees that $\sup_{0\leq t\leq T} \mathbb{E}[|f^{(k)}(X_t)|^p] < +\infty$ for all $p \geq 2$ and $k = 1, \ldots, 11$, which is necessary for the proof.

The next proposition provides useful sufficient conditions for a Gaussian process X to belong to the class $\mathcal{C}$.

Proposition 6. *Let X be a centered Gaussian process. Assume that, for all $\pi \in \mathcal{D}$ and $|\pi|$ small enough, we have $\alpha_\pi \leq C|\pi|^\alpha$, $\beta_\pi \leq C|\pi|^\beta$, $\gamma_\pi \leq C|\pi|^\gamma$, $\delta_\pi \leq C|\pi|^\delta$, where $\alpha, \delta > 0$ and β and $\gamma \in \mathbb{R}$. If $\delta > 1/3$, $\gamma > (-2\alpha) \vee (-2/3 - \alpha)$ and $\beta > (-\alpha) \vee (2/3 - 2\alpha)$ then X belongs to $\mathcal{C}$.*

Proof. Note that for $|\pi|$ small enough, $\varepsilon_\pi \leq C|\pi|^{\delta-1}$. Then, $\Psi_\pi \leq C|\pi|^{3\delta-1}$, which converges to zero if $\delta > 1/3$. On the other hand, one has that

$$\Lambda_\pi \leq C\left(|\pi|^{2\alpha+\gamma} + |\pi|^{2\alpha+\beta+\delta-1} + |\pi|^{\alpha+2\delta+\gamma} + |\pi|^{\alpha+3\delta+\beta-1}\right),$$

which converges to zero if $2\alpha+\gamma > 0$, $2\alpha+\beta+\delta-1 > 0$, $\alpha+2\delta+\gamma > 0$, $\alpha+3\delta+\beta-1 > 0$. The fact that $\delta > 1/3$ yields the sufficient conditions stated in the proposition. ■

The constant γ in the previous proposition is related to the decay of the covariance between increments. Also, the variance of the increments is controlled by α, and δ and β are related with the modulus of continuity of the covariance function in one parameter.

4. Technical lemmas

To derive the convergence results used in the proof of the main theorem, we will make use of a technical lemma based on the integration by parts formula of the Malliavin calculus:

$$\mathbb{E}[FX(h)] = \mathbb{E}\left[\langle DF, h\rangle_{\mathcal{H}}\right], \tag{4.7}$$

for any $F \in \mathbb{D}^{1,p}$ and $h \in \mathcal{H}$, $p > 1$.

We will also make use of the following result about the scalar product of iterated derivatives, which has a trivial proof.

Lemma 7. *Let $F \in \mathbb{D}^{m+n,p}$, $p > 1$, $m, n \in \mathbb{N}$, and $h_1, \ldots, h_m, g_1, \ldots, g_n \in \mathcal{H}$. Then*

$$\begin{aligned}&\left\langle D^n\left\langle D^m F, h_1 \odot \cdots \odot h_m\right\rangle_{\mathcal{H}^{\otimes m}}, g_1 \odot \cdots \odot g_n\right\rangle_{\mathcal{H}^{\otimes n}}\\&\quad = \left\langle D^{m+n}F, h_1 \odot \cdots \odot h_m \odot g_1 \odot \cdots \odot g_n\right\rangle_{\mathcal{H}^{\otimes m+n}}.\end{aligned}$$

Recall the binomial formula for the derivative operator. If $F, G \in \mathbb{D}^{m,p}$, $p > 1$, $m \in \mathbb{N}$, then

$$D^m(FG) = \sum_{k=0}^{m} \binom{m}{k} D^k F \odot D^{m-k} G. \tag{4.8}$$

Iterating the integration by parts formula (4.7) we can obtain the following useful formulas.

Lemma 8. *Let $F \in \mathbb{D}^{6,p}$, $p > 1$, and $h, g \in \mathcal{H}$. Then*

(i) $\mathbb{E}[X(h)^2 F] = \langle h, h\rangle_{\mathcal{H}}\, \mathbb{E}[F] + \mathbb{E}\left[\langle D^2 F, h^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}\right]$,
(ii) $\mathbb{E}[X(h)^3 F] = 3\,\langle h, h\rangle_{\mathcal{H}}\, \mathbb{E}\left[\langle DF, h\rangle_{\mathcal{H}}\right] + \mathbb{E}\left[\langle D^3 F, h^{\odot 3}\rangle_{\mathcal{H}^{\otimes 3}}\right]$,
(iii)

$$\begin{aligned}
\mathbb{E}[FX(h)^3X(g)^3] &= 9\,\langle h, h\rangle_{\mathcal{H}}\,\langle h, g\rangle_{\mathcal{H}}\,\langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[F] + 6\,\langle h, g\rangle_{\mathcal{H}}^3\,\mathbb{E}[F] \\
&\quad + 9\,\langle h, h\rangle_{\mathcal{H}}\,\langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^2F, h\odot g\rangle_{\mathcal{H}^{\otimes 2}}] + 9\,\langle h, h\rangle_{\mathcal{H}}\,\langle h, g\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^2F, g^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}] \\
&\quad + 9\,\langle h, g\rangle_{\mathcal{H}}\,\langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^2F, h^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}] + 18\,\langle h, g\rangle_{\mathcal{H}}^2\,\mathbb{E}[\langle D^2F, h\odot g\rangle_{\mathcal{H}^{\otimes 2}}] \\
&\quad + 3\,\langle h, h\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^4F, h\odot g^{\odot 3}\rangle_{\mathcal{H}^{\otimes 4}}] + 3\,\langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^4F, h^{\odot 3}\odot g\rangle_{\mathcal{H}^{\otimes 4}}] \\
&\quad + 9\,\langle h, g\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^4F, h^{\odot 2}\odot g^{\odot 2}\rangle_{\mathcal{H}^{\otimes 4}}] + \mathbb{E}[\langle D^6F, h^{\odot 3}\odot g^{\odot 3}\rangle_{\mathcal{H}^{\otimes 6}}].
\end{aligned}$$

Proof. Fix $k \in \mathbb{N}$; using the integration by parts formula (4.7) yields

$$\begin{aligned}
\mathbb{E}[X(h)^k F] &= \mathbb{E}[\langle D(X(h)^{k-1}F), h\rangle_{\mathcal{H}}] \\
&= (k-1)\,\langle h, h\rangle_{\mathcal{H}}\,\mathbb{E}[X(h)^{k-2}F] + \mathbb{E}[X(h)^{k-1}\langle DF, h\rangle_{\mathcal{H}}]. \qquad (4.9)
\end{aligned}$$

Combining Lemma 7, the integration by parts formula (4.7) and Eq. (4.9) we obtain (i) and (ii). To prove (iii), we can use (ii) replacing F by $FX(g)^3$ and obtain

$$\begin{aligned}
\mathbb{E}[FX(h)^3X(g)^3] &= 3\,\langle h, h\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D(FX(g)^3), h\rangle_{\mathcal{H}}] + \mathbb{E}[\langle D^3(FX(g)^3), h^{\odot 3}\rangle_{\mathcal{H}^{\otimes 3}}] \\
&= A + B.
\end{aligned}$$

We have

$$\mathbb{E}[\langle D(FX(g)^3), h\rangle_{\mathcal{H}}] = \mathbb{E}[X(g)^3\,\langle DF, h\rangle_{\mathcal{H}}] + 3\,\langle h, g\rangle_{\mathcal{H}}\,\mathbb{E}[FX(g)^2],$$

by (ii) we obtain

$$\mathbb{E}[X(g)^3\,\langle DF, h\rangle_{\mathcal{H}}] = 3\,\langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^2F, h\odot g\rangle_{\mathcal{H}^{\otimes 2}}] + \mathbb{E}[\langle D^4F, h\odot g^{\odot 3}\rangle_{\mathcal{H}^{\otimes 4}}],$$

and (i) implies

$$\begin{aligned}
\mathbb{E}[FX(g)^2] &= \langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[F] + \mathbb{E}\left[X(g)\,\langle DF, g\rangle_{\mathcal{H}}\right] \\
&= \langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[F] + \mathbb{E}[\langle D^2F, g^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}].
\end{aligned}$$

Hence, we obtain the following expression for the term A:

$$\begin{aligned}
A &= 9\,\langle h, h\rangle_{\mathcal{H}}\,\langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^2F, h\odot g\rangle_{\mathcal{H}^{\otimes 2}}] + 3\,\langle h, h\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^4F, h\odot g^{\odot 3}\rangle_{\mathcal{H}^{\otimes 4}}] \\
&\quad + 9\,\langle h, h\rangle_{\mathcal{H}}\,\langle h, g\rangle_{\mathcal{H}}\,\langle g, g\rangle_{\mathcal{H}}\,\mathbb{E}[F] + 9\,\langle h, h\rangle_{\mathcal{H}}\,\langle h, g\rangle_{\mathcal{H}}\,\mathbb{E}[\langle D^2F, g^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}].
\end{aligned}$$

For the term B, using the binomial formula, see (4.8), one obtains

$$\begin{aligned}
D^3\left(FX(g)^3\right) &= \sum_{k=0}^{3}\binom{3}{k} D^kF\odot D^{3-k}(X(g)^3) \\
&= 6Fg^{\odot 3} + 18X(g)\,DF\odot g^{\odot 2} + 9X(g)^2D^2F\odot g + X(g)^3D^3F.
\end{aligned}$$

Therefore,

$$\begin{aligned}
B &= 6\,\langle h, g\rangle_{\mathcal{H}^{\otimes 3}}^3\,\mathbb{E}[F] + 18\,\langle h, g\rangle_{\mathcal{H}}^2\,\mathbb{E}\left[X(g)\,\langle DF, h\rangle_{\mathcal{H}}\right] \\
&\quad + 9\,\langle h, g\rangle_{\mathcal{H}}\,\mathbb{E}[X(g)^2\langle D^2F, h^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}] + \mathbb{E}[X(g)^3\langle D^3F, h^{\odot 3}\rangle_{\mathcal{H}^{\otimes 3}}]. \qquad (4.10)
\end{aligned}$$

We have that $\mathbb{E}[X(g)\langle DF, h\rangle_{\mathcal{H}}] = \mathbb{E}[\langle D^2F, h \odot g\rangle_{\mathcal{H}^{\otimes 2}}]$. For the expectation in the third summand of Eq. (4.10), (i) yields

$$\mathbb{E}[X(g)^2 \langle D^2F, h^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}] = \langle g, g\rangle_{\mathcal{H}}\, \mathbb{E}[\langle D^2F, h^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}] + \mathbb{E}[\langle D^4F, h^{\odot 2} \odot g^{\odot 2}\rangle_{\mathcal{H}^{\otimes 4}}],$$

and for the fourth term (ii) leads to

$$\begin{aligned}&\mathbb{E}[(X(g))^3 \langle D^3F, h^{\odot 3}\rangle_{\mathcal{H}^{\otimes 3}}] \\ &\quad = 3\langle g, g\rangle_{\mathcal{H}}\, \mathbb{E}[\langle D^4F, h^{\odot 3} \odot g\rangle_{\mathcal{H}^{\otimes 4}}] + \mathbb{E}[\langle D^6F, h^{\odot 3} \odot g^{\odot 3}\rangle_{\mathcal{H}^{\otimes 6}}].\end{aligned}$$

Hence,

$$\begin{aligned}B &= 6\langle h, g\rangle_{\mathcal{H}}^3\, \mathbb{E}[F] + 18\langle h, g\rangle_{\mathcal{H}}^2\, \mathbb{E}[\langle D^2F, h \odot g\rangle_{\mathcal{H}^{\otimes 2}}] \\ &\quad + 9\langle h, g\rangle_{\mathcal{H}} \langle g, g\rangle_{\mathcal{H}}\, \mathbb{E}[\langle D^2F, h^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}] + 9\langle h, g\rangle_{\mathcal{H}}\, \mathbb{E}[\langle D^4F, h^{\odot 2} \odot g^{\odot 2}\rangle_{\mathcal{H}^{\otimes 4}}] \\ &\quad + 3\langle g, g\rangle_{\mathcal{H}}\, \mathbb{E}[\langle D^4F, h^{\odot 3} \odot g\rangle_{\mathcal{H}^{\otimes 4}}] + \mathbb{E}[\langle D^6F, h^{\odot 3} \odot g^{\odot 3}\rangle_{\mathcal{H}^{\otimes 6}}].\end{aligned}$$

Adding A and B we conclude the proof. ∎

Lemma 9. *If $f \in \mathcal{C}^m_{l\text{-}\exp}(\mathbb{R})$, $m, l \in \mathbb{N}$, there exists a finite constant $a_{T,l}$ such that for all $k_1, \ldots, k_l = 0, \ldots, m$, and random variables ξ_i, $i = 1, \ldots, l$ satisfying $|\xi_i| \le \sup_{t\in[0,T]} |X_t|$ we have $\mathbb{E}[|f^{(k_1)}(\xi_1)\cdots f^{(k_l)}(\xi_l)|] \le a_{T,l}$.*

Proof. We have that

$$\begin{aligned}\mathbb{E}[|f^{(k_1)}(\xi_1)\cdots f^{(k_l)}(\xi_l)|] &\le C_T^l \mathbb{E}\left[\exp\left\{c_T \sum_{i=1}^{l} \xi_i^2\right\}\right] \\ &\le C_T^l \mathbb{E}\left[\exp\left\{l c_T \sup_{0\le t\le T} X_t^2\right\}\right] =: a_{T,l},\end{aligned}$$

which is finite by (2.3). ∎

5. Convergence results

In this section we prove the convergence results used in the proof of the main theorem. We will make use of the notation of Section 3, and in addition we set $\delta_i = \mathbf{1}_{(t_i,t_{i+1}]}$ and $\sigma_i = (\mathbf{1}_{[0,t_{i+1}]} + \mathbf{1}_{[0,t_i]})/2$.

Proposition 10. *We have* $\lim_{|\pi|\to 0, \pi\in\mathcal{D}} \mathbb{E}[(R_3^\pi)^2] = 0$.

Proof. Set $F_{i,j} = f^{(3)}(S_iX)f^{(3)}(S_jX)$. Then, Lemma 8 gives

$$\begin{aligned}\mathbb{E}[(R_3^\pi)^2] &= \sum_{i,j=0}^{n-1} \mathbb{E}[F_{i,j} X(\delta_i)^3 X(\delta_j)^3] \\ &= 9A_1 + 6A_2 + 9A_3 + 9A_4 + 9A_5 + 18A_6 + 3A_7 + 3A_8 + 9A_9 + A_{10},\end{aligned}$$

where

$$A_1 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_i, \delta_i\rangle_{\mathcal{H}} \langle \delta_i, \delta_j\rangle_{\mathcal{H}} \langle \delta_j, \delta_j\rangle_{\mathcal{H}}\, \mathbb{E}\left[F_{i,j}\right],$$

$$A_2 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_i, \delta_j \rangle_{\mathcal{H}}^3 \mathbb{E}\left[F_{i,j}\right],$$

$$A_3 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_i, \delta_i \rangle_{\mathcal{H}} \langle \delta_j, \delta_j \rangle_{\mathcal{H}} \mathbb{E}[\langle D^2 F_{i,j}, \delta_i \odot \delta_j \rangle_{\mathcal{H}^{\otimes 2}}],$$

$$A_4 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_i, \delta_i \rangle_{\mathcal{H}} \langle \delta_i, \delta_j \rangle_{\mathcal{H}} \mathbb{E}[\langle D^2 F_{i,j}, \delta_j^{\odot 2} \rangle_{\mathcal{H}^{\otimes 2}}],$$

$$A_5 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_i, \delta_j \rangle_{\mathcal{H}} \langle \delta_j, \delta_j \rangle_{\mathcal{H}} \mathbb{E}[\langle D^2 F_{i,j}, \delta_i^{\odot 2} \rangle_{\mathcal{H}^{\otimes 2}}],$$

$$A_6 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_i, \delta_j \rangle_{\mathcal{H}}^2 \mathbb{E}[\langle D^2 F_{i,j}, \delta_i \odot \delta_j \rangle_{\mathcal{H}^{\otimes 2}}],$$

$$A_7 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_i, \delta_i \rangle_{\mathcal{H}} \mathbb{E}[\langle D^4 F_{i,j}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}],$$

$$A_8 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_j, \delta_j \rangle_{\mathcal{H}} \mathbb{E}[\langle D^4 F_{i,j}, \delta_i^{\odot 3} \odot \delta_j \rangle_{\mathcal{H}^{\otimes 4}}],$$

$$A_9 \triangleq \sum_{i,j=0}^{n-1} \langle \delta_i, \delta_j \rangle_{\mathcal{H}} \mathbb{E}[\langle D^4 F_{i,j}, \delta_i^{\odot 2} \odot \delta_j^{\odot 2} \rangle_{\mathcal{H}^{\otimes 4}}],$$

$$A_{10} \triangleq \sum_{i,j=0}^{n-1} \mathbb{E}[\langle D^6 F_{i,j}, \delta_i^{\odot 3} \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 6}}].$$

To check the convergence of these terms we will make use of Lemma 9, and the Cauchy–Schwartz and Hölder inequalities.

- *Term* A_1: One has

$$|A_1| = \left| \sum_{i,j=0}^{n-1} \mathbb{E}[(\Delta_i X)^2] \mathbb{E}[(\Delta_j X)^2] \mathbb{E}\left[\Delta_i X \Delta_j X\right] \mathbb{E}[f^{(3)}(S_i X) f^{(3)}(S_j X)] \right|$$

$$\leq a_{T,2} (\sup_{i \geq 0} \mathbb{E}[(\Delta_i X)^2])^2 \sum_{i,j=0}^{n-1} \left| \mathbb{E}\left[\Delta_i X \Delta_j X\right] \right| \leq a_{T,2} \alpha_\pi^2 \gamma_\pi,$$

  which converges to 0 as $|\pi| \to 0$, $\pi \in \mathcal{D}$, by condition (2.5).
- *Term* A_2: This term can be treated as term A_1 because

$$\left| \langle \delta_i, \delta_j \rangle_{\mathcal{H}} \right|^3 = \left| \langle \delta_i, \delta_j \rangle_{\mathcal{H}} \right|^2 \left| \langle \delta_i, \delta_j \rangle_{\mathcal{H}} \right| \leq \left| \langle \delta_i, \delta_i \rangle_{\mathcal{H}} \right| \left| \langle \delta_j, \delta_j \rangle_{\mathcal{H}} \right| \left| \langle \delta_i, \delta_j \rangle_{\mathcal{H}} \right|.$$

- *Term* A_3: We have

$$|A_3| = \left| \sum_{i,j=0}^{n-1} \mathbb{E}[(\Delta_i X)^2] \mathbb{E}[(\Delta_j X)^2] \sum_{k=0}^{2} \binom{2}{k} \right.$$

$$\left. \times \mathbb{E}[f^{(3+k)}(S_i X) f^{(5-k)}(S_j X)] \langle \sigma_i^{\odot k} \odot \sigma_j^{\odot 2-k}, \delta_i \odot \delta_j \rangle_{\mathcal{H}^{\otimes 2}} \right|$$

$$\leq a_{T,2}(\sup_{i\geq 0}\mathbb{E}[(\Delta_i X)^2])^2\sum_{i,j=0}^{n-1}\{|\langle\sigma_j^{\odot 2},\delta_i\odot\delta_j\rangle_{\mathcal{H}^{\otimes 2}}| + 2\left|\langle\sigma_i\odot\sigma_j,\delta_i\odot\delta_j\rangle_{\mathcal{H}^{\otimes 2}}\right| + |\langle\sigma_i^{\odot 2},\delta_i\odot\delta_j\rangle_{\mathcal{H}^{\otimes 2}}|\}.$$

Thus it suffices to check the convergence to 0 of the terms

$$A_3^1 \triangleq \alpha_\pi^2\sum_{i,j=0}^{n-1}|\langle\sigma_j^{\odot 2},\delta_i\odot\delta_j\rangle_{\mathcal{H}^{\otimes 2}}|,\qquad A_3^2 \triangleq \alpha_\pi^2\sum_{i,j=0}^{n-1}|\langle\sigma_i\odot\sigma_j,\delta_i\odot\delta_j\rangle_{\mathcal{H}^{\otimes 2}}|,$$

$$A_3^3 \triangleq \alpha_\pi^2\sum_{i,j=0}^{n-1}|\langle\sigma_i^{\odot 2},\delta_i\odot\delta_j\rangle_{\mathcal{H}^{\otimes 2}}|.$$

By the definition of the scalar products we have

$$\left\langle\sigma_j^{\odot 2},\delta_i\odot\delta_j\right\rangle_{\mathcal{H}^{\otimes 2}} = \langle\sigma_j,\delta_i\rangle_{\mathcal{H}}\left\langle\sigma_j,\delta_j\right\rangle_{\mathcal{H}},$$

$$\left\langle\sigma_i\odot\sigma_j,\delta_i\odot\delta_j\right\rangle_{\mathcal{H}^{\otimes 2}} = \frac{1}{2}(\langle\sigma_i,\delta_i\rangle_{\mathcal{H}}\left\langle\sigma_j,\delta_j\right\rangle_{\mathcal{H}} + \left\langle\sigma_i,\delta_j\right\rangle_{\mathcal{H}}\left\langle\sigma_j,\delta_i\right\rangle_{\mathcal{H}}),$$

$$\left\langle\sigma_i^{\odot 2},\delta_i\odot\delta_j\right\rangle_{\mathcal{H}^{\otimes 2}} = \langle\sigma_i,\delta_i\rangle_{\mathcal{H}}\left\langle\sigma_i,\delta_j\right\rangle_{\mathcal{H}}.$$

Therefore, the convergence to 0 of A_3^1, A_3^2 and A_3^3 reduces to showing the convergence to 0 of $A_3^{k_1,k_2} \triangleq \alpha_\pi^2\sum_{i,j=0}^{n-1}|\langle\sigma_{k_1},\delta_i\rangle_{\mathcal{H}}||\langle\sigma_{k_2},\delta_j\rangle_{\mathcal{H}}|$, where $k_1,k_2\in\{i,j\}$. Notice that $\sup_k|\langle\sigma_k,\delta_i\rangle_{\mathcal{H}}|\leq\sup_{0\leq t\leq T}|\mathbb{E}[X_t\,(\Delta_i X)]|$. As a consequence,

$$A_3^{k_1,k_2}\leq\alpha_\pi^2\sup_{0\leq t\leq T}\left(\sum_{j=0}^{n-1}\left|\mathbb{E}[X_t\Delta_j X]\right|\right)\sum_{i=0}^{n-1}\sup_{0\leq t\leq T}|\mathbb{E}[X_t\Delta_i X]| = \alpha_\pi^2\beta_\pi\varepsilon_\pi,$$

which converges to 0 as $|\pi|\to 0$, $\pi\in\mathcal{D}$, by condition (2.5).

- *Terms $A_4 = A_5$*: We have

$$|A_4| = \left|\sum_{i,j=0}^{n-1}\langle\delta_i,\delta_i\rangle_{\mathcal{H}}\left\langle\delta_i,\delta_j\right\rangle_{\mathcal{H}}\mathbb{E}[\langle D^2F_{i,j},\delta_i^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}]\right|$$

$$= \left|\sum_{i,j=0}^{n-1}\langle\delta_i,\delta_i\rangle_{\mathcal{H}}\left\langle\delta_i,\delta_j\right\rangle_{\mathcal{H}}\sum_{k=0}^{2}\binom{2}{k}\times\mathbb{E}[f^{(3+k)}(S_iX)\,f^{(5-k)}(S_jX)]\langle\sigma_i^{\odot k}\odot\sigma_j^{\odot 2-k},\delta_i^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}\right|.$$

The estimate $|\langle\sigma_i^{\odot k}\odot\sigma_j^{\odot 2-k},\delta_i^{\odot 2}\rangle_{\mathcal{H}^{\otimes 2}}|\leq C\sup_{0\leq t\leq T}|\mathbb{E}\,[X_t\Delta_i X]|^2$ yields

$$A_4\leq Ca_{T,2}\sum_{i,j=0}^{n-1}\mathbb{E}[(\Delta_i X)^2]\left|\mathbb{E}\left[\Delta_i X\Delta_j X\right]\right|\sup_{0\leq t\leq T}\left|\mathbb{E}\left[X_t\Delta_j X\right]\right|^2$$

$$\leq Ca_{T,2}\sup_{i\geq 0}\mathbb{E}[(\Delta_i X)^2](\sup_{\substack{0\leq t\leq T\\ i\geq 0}}|\mathbb{E}\,[X_t\Delta_i X]|)^2\sum_{i,j=0}^{n-1}\left|\mathbb{E}\left[\Delta_i X\Delta_j X\right]\right|$$

$$\leq Ca_{T,2}\alpha_\pi\delta_\pi^2\gamma_\pi,$$

which converges to 0 as $|\pi|\to 0$, $\pi\in\mathcal{D}$, by condition (2.5).

- *Term* A_6: Using that $\langle \delta_i, \delta_j \rangle_{\mathcal{H}}^2 \le \langle \delta_i, \delta_i \rangle_{\mathcal{H}} \langle \delta_j, \delta_j \rangle_{\mathcal{H}}$, the convergence to 0 of this term follows from the convergence of A_3.
- *Term* $A_7 = A_8$: We have

$$
\begin{aligned}
|A_7| &= \sum_{i,j=0}^{n-1} \left| \mathbb{E}[(\Delta_i X)^2] \sum_{k=0}^{4} \binom{4}{k} \mathbb{E}[f^{(3+k)}(S_i X)\, f^{(7-k)}(S_j X)] \right| \\
&\quad \times \left| \left\langle \sigma_i^{\odot k} \odot \sigma_j^{\odot 4-k}, \delta_i \odot \delta_j^{\odot 3} \right\rangle_{\mathcal{H}^{\otimes 4}} \right| \\
&\le a_{T,2} \sup_{i \ge 0} \mathbb{E}[(\Delta_i X)^2] \\
&\quad \times \sum_{i,j=0}^{n-1} \{ |\langle \sigma_j^{\odot 4}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}| + 4 |\langle \sigma_i \odot \sigma_j^{\odot 3}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}| \\
&\quad + 6 |\langle \sigma_i^{\odot 2} \odot \sigma_j^{\odot 2}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}| + 4 |\langle \sigma_i^{\odot 3} \odot \sigma_j, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}| \\
&\quad + |\langle \sigma_i^{\odot 4}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}| \}.
\end{aligned}
$$

Then it suffices to check the convergence to 0 of the following terms:

$$
\begin{aligned}
A_7^1 &\triangleq \alpha_\pi \sum_{i,j=0}^{n-1} |\langle \sigma_j^{\odot 4}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}|, \\
A_7^2 &\triangleq \alpha_\pi \sum_{i,j=0}^{n-1} |\langle \sigma_i \odot \sigma_j^{\odot 3}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}|, \\
A_7^3 &\triangleq \alpha_\pi \sum_{i,j=0}^{n-1} |\langle \sigma_i^{\odot 2} \odot \sigma_j^{\odot 2}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}|, \\
A_7^4 &\triangleq \alpha_\pi \sum_{i,j=0}^{n-1} |\langle \sigma_i^{\odot 3} \odot \sigma_j, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}|, \\
A_7^5 &\triangleq \alpha_\pi \sum_{i,j=0}^{n-1} |\langle \sigma_i^{\odot 4}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}}|.
\end{aligned}
$$

It is easy to show that

$$
\begin{aligned}
&\langle \sigma_j^{\odot 4}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}} = \langle \sigma_j, \delta_i \rangle_{\mathcal{H}} \langle \sigma_j, \delta_j \rangle_{\mathcal{H}}^3, \\
&\langle \sigma_i \odot \sigma_j^{\odot 3}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}} = \frac{1}{24} (6 \langle \sigma_i, \delta_i \rangle_{\mathcal{H}} \langle \sigma_j, \delta_j \rangle_{\mathcal{H}}^3 \\
&\qquad + 18 \langle \sigma_i, \delta_j \rangle_{\mathcal{H}} \langle \sigma_j, \delta_i \rangle_{\mathcal{H}} \langle \sigma_j, \delta_j \rangle_{\mathcal{H}}^2), \\
&\langle \sigma_i^{\odot 2} \odot \sigma_j^{\odot 2}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}} = \frac{1}{24} (12 \langle \sigma_i, \delta_i \rangle_{\mathcal{H}} \langle \sigma_i, \delta_j \rangle_{\mathcal{H}} \langle \sigma_j, \delta_j \rangle_{\mathcal{H}}^2 \\
&\qquad + 12 \langle \sigma_j, \delta_j \rangle_{\mathcal{H}} \langle \sigma_j, \delta_i \rangle_{\mathcal{H}} \langle \sigma_i, \delta_j \rangle_{\mathcal{H}}^2), \\
&\langle \sigma_i^{\odot 3} \odot \sigma_j, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}} = \frac{1}{24} (6 \langle \sigma_j, \delta_i \rangle_{\mathcal{H}} \langle \sigma_i, \delta_j \rangle_{\mathcal{H}}^3 \\
&\qquad + 18 \langle \sigma_i, \delta_i \rangle_{\mathcal{H}} \langle \sigma_j, \delta_j \rangle_{\mathcal{H}} \langle \sigma_i, \delta_j \rangle_{\mathcal{H}}^2), \\
&\langle \sigma_i^{\odot 4}, \delta_i \odot \delta_j^{\odot 3} \rangle_{\mathcal{H}^{\otimes 4}} = \langle \sigma_i, \delta_i \rangle_{\mathcal{H}} \langle \sigma_i, \delta_j \rangle_{\mathcal{H}}^3.
\end{aligned}
$$

Therefore, the convergence of A_7^l, $l = 0, \ldots, 5$, to 0 reduces to showing the convergence to 0 of

$$A_7^{k_1,k_2} \triangleq \alpha_\pi \left(\sup_{k,h} |\langle \sigma_k, \delta_h \rangle_{\mathcal{H}}|^2 \right) \sum_{i,j=0}^{n-1} \left| \langle \sigma_{k_1}, \delta_i \rangle_{\mathcal{H}} \right| \left| \langle \sigma_{k_2}, \delta_j \rangle_{\mathcal{H}} \right|,$$

where $k_1, k_2 \in \{i, j\}$. Using that $\sup_{h,k} |\langle \sigma_k, \delta_h \rangle_{\mathcal{H}}| \le \sup_i \sup_{0 \le t \le T} |\mathbb{E}[X_t \Delta_i X]|$, we obtain $A_7^{k_1,k_2} \le \alpha_\pi \delta_\pi^2 \beta_\pi \varepsilon_\pi$, which converges to 0 as $|\pi| \to 0$, $\pi \in \mathcal{D}$, by condition (2.5).

- *Term* A_9: Proceeding as in the case of the term A_4, we have

$$\begin{aligned} A_9 &\le C a_{T,2} \sum_{i,j=0}^{n-1} \left| \mathbb{E}\left[\Delta_i X \Delta_j X \right] \right| \sup_{0 \le t \le T} |\mathbb{E}[X_t \Delta_i X]|^2 \sup_{0 \le t \le T} \left| \mathbb{E}\left[X_t \Delta_j X \right] \right|^2 \\ &\le C a_{T,2} \left(\sum_{i,j=0}^{n-1} \left| \mathbb{E}\left[\Delta_i X \Delta_j X \right] \right|^3 \right)^{1/3} \left(\sum_{i=0}^{n-1} \sup_{0 \le t \le T} |\mathbb{E}[X_t \Delta_i X]|^3 \right)^{4/3}. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{i,j=0}^{n-1} \left| \mathbb{E}\left[\Delta_i X \Delta_j X \right] \right|^3 &\le \sum_{i,j=0}^{n-1} \mathbb{E}[(\Delta_i X)^2] \mathbb{E}[(\Delta_j X)^2] \left| \mathbb{E}\left[\Delta_i X \Delta_j X \right] \right| \\ &\le \sup_{i \ge 0} \mathbb{E}[(\Delta_i X)^2]^2 \sum_{i,j=0}^{n-1} \left| \mathbb{E}\left[\Delta_i X \Delta_j X \right] \right| \le \alpha_\pi^2 \gamma_\pi, \end{aligned} \tag{5.11}$$

and

$$\sum_{i=0}^{n-1} \sup_{0 \le t \le T} |\mathbb{E}[X_t \Delta_i X]|^3 \le \sup_{\substack{0 \le t \le T \\ i \ge 0}} |\mathbb{E}[X_t \Delta_i X]|^2 \sum_{i=0}^{n-1} \sup_{0 \le t \le T} |\mathbb{E}[X_t \Delta_i X]|.$$

Hence, $A_9 \le C a_{T,2} \left(\alpha_\pi^2 \gamma_\pi \right)^{1/3} \left(\delta_\pi^2 \varepsilon_\pi \right)^{4/3}$, which converges to 0 as $|\pi| \to 0$, $\pi \in \mathcal{D}$, by conditions (2.5) and (2.6).

- *Term* A_{10}: By the same arguments as in the case of the term A_4, one obtains using Eq. (5.11)

$$\begin{aligned} A_{10} &\le C a_{T,2} \sum_{i,j=0}^{n-1} \sup_{0 \le t \le T} |\mathbb{E}[X_t \Delta_i X]|^3 \sup_{0 \le t \le T} \left| \mathbb{E}\left[X_t \Delta_j X \right] \right|^3 \\ &= C a_{T,2} \left(\sum_{i=0}^{n-1} \sup_{0 \le s \le T} |\mathbb{E}[X_s \Delta_i X]|^3 \right)^2 \le C a_{T,2} \left(\delta_\pi^2 \varepsilon_\pi \right)^2, \end{aligned}$$

which converges to 0 as $|\pi| \to 0$, $\pi \in \mathcal{D}$, by condition (2.6). ■

Proposition 11. *We have* $\lim_{|\pi| \to 0, \pi \in \mathcal{D}} \mathbb{E}[(R_5^\pi)^2] = 0$.

Proof. We have that

$$\sum_{i=0}^{n-1} f^{(5)}(S_i X)(\Delta_i X)^5 = \sum_{i=0}^{n-1} f^{(5)}(S_i X)(\Delta_i X)^2 (\Delta_i X)^3.$$

Now we can apply Proposition 10 replacing $f^{(3)}(S_i X)$ by $f^{(5)}(S_i X)(\Delta_i X)^2$. This can be done because $f^{(5)}(S_i X)(\Delta_i X)^2$ satisfies the exponential growth condition (2.2). However, this time the derivatives involved are up to order 11. ■

Proposition 12. *We have* $\lim_{|\pi|\to 0} \mathbb{E}[(R_6^\pi)^2] = 0$.

Proof. We prove the result for the term $\sum_{i=0}^{n-1} f^{(6)}(\widehat{S}_i^+)\,(\Delta_i X)^6$, and the proof of the others is identical. We have

$$
\begin{aligned}
\left(\mathbb{E}[(R_6^\pi)^2]\right)^{1/2} &= \left\| \sum_{i=0}^{n-1} f^{(6)}(\widehat{S}_i^+)\,(\Delta_i X)^6 \right\|_{L^2(\Omega)} \\
&\le \sum_{i=0}^{n-1} \left\| f^{(6)}(\widehat{S}_i^+)\,(\Delta_i X)^6 \right\|_{L^2(\Omega)} \le \sum_{i=0}^{n-1} \left(\mathbb{E}[(f^{(6)}(\widehat{S}_i^+))^4]\right)^{1/4} \left(\mathbb{E}[(\Delta_i X)^{24}]\right)^{1/4} \\
&\le \left(a_{T,4}\right)^{1/4} \sum_{i=0}^{n-1} \left(\mathbb{E}[(\Delta_i X)^{24}]\right)^{6/24} = \left(a_{T,4}\right)^{1/4} \kappa\,(24)^6 \sum_{i=0}^{n-1} \left(\mathbb{E}[(\Delta_i X)^2]\right)^3 \\
&\le \left(a_{T,4}\right)^{1/4} \kappa\,(24)^6 \left(\sup_{i\ge 0} \mathbb{E}[(\Delta_i X)^2]\right)^2 \sum_{i=0}^{n-1} \mathbb{E}[(\Delta_i X)^2],
\end{aligned}
$$

where we have used the triangle inequality, Cauchy–Schwartz inequality, Lemma 9, and that for all $p > 0$ if ξ is a centered Gaussian variable one has $\|\xi\|_{L^p(\Omega)} = \kappa\,(p)\,\|\xi\|_{L^2(\Omega)}$, where $\kappa\,(p) = \sqrt{2}\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}\right)^{1/p}$, $p > 0$. Finally, the result follows from condition (2.5). ■

6. An example

In this section we will provide an example of a Gaussian process belonging to the class $\mathcal{C}$. As in the previous sections, we fix a class $\mathcal{D}$ of partitions $\pi = \{0 = t_0 < \cdots < t_n = T\}$ of $[0, T]$ satisfying condition (2.4).

The bifractional Brownian motion $B^{H,K} = \{B_t^{H,K}, t \ge 0\}, 0 < H < 1, 0 < K \le 1$, is a centered Gaussian process with the covariance function

$$
\mathbb{E}\left[B_s^{H,K} B_t^{H,K}\right] = R_{H,K}\,(s,t) = \frac{1}{2^K}\left\{\left(s^{2H} + t^{2H}\right)^K - |s-t|^{2HK}\right\}. \tag{6.12}
$$

If we take $K = 1$, then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter H. The following property is shown in [10]:

$$
2^{-K}\,|t-s|^{2HK} \le \mathbb{E}\left[\left(B_t^{H,K} - B_s^{H,K}\right)^2\right] \le 2^{1-K}\,|t-s|^{2HK}. \tag{6.13}
$$

In [15] Russo and Tudor have established an Itô–Stratonovich formula for this process using the regularization approach.

Recall the following inequality, if $\alpha > 0$ and $0 < y < x < T$ then

$$
x^\alpha - y^\alpha \le C\,(x-y)^\alpha, \tag{6.14}
$$

where constant C only depends on T and can be taken equal to 1 if $\alpha \in (0, 1]$. We start with some technical lemmas.

Lemma 13. *Set* $\gamma_\pi = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left|\mathbb{E}[\Delta_i B^{H,K}\Delta_j B^{H,K}]\right|$. *Then*

$$\gamma_\pi \le \begin{cases} C(1+\mathbf{1}_{\{K\neq 1\}}|\pi|^{2HK-1}) & \text{if } HK>1/2 \\ C|\pi|^{2HK-1}(\mathbf{1}_{\{K\neq 1\}}+\ln(|\pi|^{-1})) & \text{if } HK=1/2 \\ C|\pi|^{2HK-1} & \text{if } 1/6<HK<1/2. \end{cases} \tag{6.15}$$

Proof. Using (6.12) one obtains

$$\begin{aligned}
&\sum_{i,j=0}^{n-1}\left|\mathbb{E}\left[\left(\Delta_i B^{H,K}\right)\left(\Delta_j B^{H,K}\right)\right]\right| \\
&= \sum_{i,j=0}^{n-1}\left|R_{H,K}\left(t_{i+1},t_{j+1}\right)+R_{H,K}\left(t_i,t_j\right)-R_{H,K}\left(t_{i+1},t_j\right)-R_{H,K}\left(t_i,t_{j+1}\right)\right| \\
&\le \frac{1}{2^{2K}}\sum_{i,j=0}^{n-1}\left|\left|t_{i+1}-t_j\right|^{2HK}+\left|t_i-t_{j+1}\right|^{2HK}-\left|t_{i+1}-t_{j+1}\right|^{2HK}-\left|t_i-t_j\right|^{2HK}\right| \\
&\quad+\frac{1}{2^{2K}}\sum_{i,j=0}^{n-1}\left|\left(t_{i+1}^{2H}+t_{j+1}^{2H}\right)^K-\left(t_{i+1}^{2H}+t_j^{2H}\right)^K+\left(t_i^{2H}+t_j^{2H}\right)^K-\left(t_i^{2H}+t_{j+1}^{2H}\right)^K\right| \\
&=: \frac{1}{2^{2K}}\left(A_\pi+B_\pi\right).
\end{aligned}$$

The same ideas as were used in the proof of Proposition 13 in [14] give

$$A_\pi \le \begin{cases} C & \text{if } HK>1/2 \\ C|\pi|^{2HK-1}\ln\left(|\pi|^{-1}\right) & \text{if } HK=1/2 \\ C|\pi|^{2HK-1} & \text{if } 1/6<HK<1/2. \end{cases}$$

On the other hand, if $K=1$ the term B_π vanishes; otherwise define

$$\varphi_j(t) := \left(t^{2H}+t_{j+1}^{2H}\right)^K-\left(t^{2H}+t_j^{2H}\right)^K, \quad t>0. \tag{6.16}$$

It is easy to show that $\varphi_j(t)$ is positive and nonincreasing. Furthermore, $\sum_{j=0}^{n-1}\varphi_j(t) = \left(t^{2H}+t_n^{2H}\right)^K-\left(t^{2H}+t_0^{2H}\right)^K$. Therefore, one has

$$\begin{aligned}
B_\pi &= \sum_{i,j=0}^{n-1}\left|\varphi_j(t_{i+1})-\varphi_j(t_i)\right| = \sum_{i,j=0}^{n-1}\varphi_j(t_i)-\varphi_j(t_{i+1}) \\
&= \sum_{i=0}^{n-1}\left(t_i^{2H}+t_n^{2H}\right)^K-\left(t_i^{2H}+t_0^{2H}\right)^K-\left(t_{i+1}^{2H}+t_n^{2H}\right)^K+\left(t_{i+1}^{2H}+t_0^{2H}\right)^K \\
&\le \sum_{i=0}^{n-1}\left|\left(t_{i+1}^{2H}+t_0^{2H}\right)^K-\left(t_i^{2H}+t_0^{2H}\right)^K\right|+\sum_{i=0}^{n-1}\left|\left(t_{i+1}^{2H}+t_n^{2H}\right)^K-\left(t_i^{2H}+t_n^{2H}\right)^K\right| \\
&\le 2\sum_{i=0}^{n-1}\left(t_{i+1}^{2H}-t_i^{2H}\right)^K \le C\sum_{i=0}^{n-1}(t_{i+1}-t_i)^{2HK} \le C|\pi|^{2HK-1},
\end{aligned}$$

where we have used the inequality (6.14). Taking into account the bounds obtained for A_π and B_π, we can conclude the proof. ■

Lemma 14. *If $\pi = \{0 = t_0 < \cdots < t_n = T\}$ is a partition of $[0, T]$ and*

$$\beta_\pi = \sup_{0\le t\le T}\left(\sum_{i=0}^{n-1}\left|\mathbb{E}[B_t^{H,K}\Delta_i B^{H,K}]\right|\right),$$

then

$$\beta_\pi \le C, \tag{6.17}$$

where C only depends on T.

Proof. From (6.12) one obtains

$$\begin{aligned}
&\left|\mathbb{E}[B_t^{H,K}\Delta_i B^{H,K}]\right| \\
&= \frac{1}{2^K}\left|(t^{2H}+t_{i+1}^{2H})^K - (t^{2H}+t_i^{2H})^K + |t-t_i|^{2HK} - |t-t_{i+1}|^{2HK}\right| \\
&\le \frac{1}{2^K}\left(\varphi_i(t) + \left||t-t_i|^{2HK} - |t-t_{i+1}|^{2HK}\right|\right),
\end{aligned}$$

where $\varphi_i(t)$ is defined in (6.16). Notice that

$$\sup_{0\le t\le T}\sum_{i=0}^{n-1}\varphi_i(t) \le \sum_{i=0}^{n-1}\sup_{0\le t\le T}\varphi_i(t) = \sum_{i=0}^{n-1}\varphi_i(0) = T^{2HK} < +\infty.$$

Define $\xi_i(t) := |t-t_i|^{2HK} - |t-t_{i+1}|^{2HK}$. One has for $k_1 < k_2$

$$\sum_{i=k_1}^{k_2}\xi_i(t) = \left|t-t_{k_1}\right|^{2HK} - \left|t-t_{k_2+1}\right|^{2HK}. \tag{6.18}$$

Clearly, if $t \le t_i$ then $\xi_i(t) < 0$ and if $t \ge t_{i+1}$ then $\xi_i(t) > 0$. Furthermore, for each $t \in [0, T]$ there exist $k(t) \in \{0, \ldots, n-1\}$ such that $t \in [t_{k(t)}, t_{k(t)+1}]$. Then

$$\begin{aligned}
\sum_{i=0}^{n-1}|\xi_i(t)| &= \sum_{i=0}^{k(t)-1}\xi_i(t) + \left|\xi_{k(t)}\right| - \sum_{i=k(t)+1}^{n-1}\xi_i(t) \\
&= |t|^{2HK} - \left|t-t_{k(t)}\right|^{2HK} + \left|\left|t-t_{k(t)}\right|^{2HK} - \left|t-t_{k(t)+1}\right|^{2HK}\right| \\
&\quad + |t-T|^{2HK} - \left|t-t_{k(t)+1}\right|^{2HK} \le CT^{2HK},
\end{aligned}$$

where we have used (6.18), and the result follows. ■

Now we are ready to state the main result of this section.

Proposition 15. *The bifractional Brownian motion $B_t^{H,K}$ with $1/6 < HK < 1$ belongs to the class $\mathcal{C}$, and, hence, Theorem 4 holds for this process.*

Proof. From Eq. (6.13) it follows that

$$\mathbb{E}[(\Delta_i B^{H,K})^2] \le C\left|t_{i_{+1}} - t_i\right|^{2HK},$$

and hence

$$\alpha_\pi = \sup_{i\geq 0} \mathbb{E}[(\Delta_i B^{H,K})^2] \leq C\,|\pi|^{2HK}. \tag{6.19}$$

We have

$$\left|\mathbb{E}[B_t^{H,K} \Delta_i B^{H,K}]\right| = \frac{1}{2^K}\Big|(t^{2H}+t_{i+1}^{2H})^K - (t^{2H}+t_i^{2H})^K + |t-t_i|^{2HK} - |t-t_{i+1}|^{2HK}\Big|.$$

Using that

$$(t^{2H}+t_{i+1}^{2H})^K - (t^{2H}+t_i^{2H})^K \leq \begin{cases} (t_{i+1}-t_i)^{2HK} & \text{if } 1/6 < HK \leq 1/2 \\ C\,(t_{i+1}-t_i) & \text{if } HK > 1/2, \end{cases}$$

and

$$|t-t_i|^{2HK} - |t-t_{i+1}|^{2HK} \leq \begin{cases} (t_{i+1}-t_i)^{2HK} & \text{if } 1/6 < HK \leq 1/2 \\ C\,(t_{i+1}-t_i) & \text{if } HK > 1/2, \end{cases}$$

which follows from inequality (6.14) and the mean value theorem, we get

$$\delta_\pi = \sup_{\substack{0\leq t\leq T \\ i\geq 0}} \left|\mathbb{E}[B_t^{H,K}\Delta_i B^{H,K}]\right| \leq \begin{cases} |\pi|^{2HK} & \text{if } 1/6 < HK \leq 1/2 \\ C\,|\pi| & \text{if } HK > 1/2 \end{cases} \tag{6.20}$$

and

$$\varepsilon_\pi = \sum_{i=0}^{n-1} \sup_{0\leq t\leq T} \left|\mathbb{E}[B_t^{H,K}\Delta_i B^{H,K}]\right| \leq \begin{cases} C\,|\pi|^{2HK-1} & \text{if } 1/6 < HK \leq 1/2 \\ C & \text{if } HK > 1/2. \end{cases} \tag{6.21}$$

Combining the estimates (6.15), (6.17), (6.19)–(6.21) and Proposition 6 we obtain the result. ■

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